Product Square Cordial Labeling of Some Graphs

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ABSTRACT

A product square cordial labeling of a graph G with vertex set V and edge set E is a bijection f from $V(G)$ to $\{1,2,\ldots,|V(G)|\}$ if there exists an induced edge function f^* from $E(G)$ to $\{0,1\}$ satisfies the following conditions. (i) For each edge $uv \in E(G)$ is assigned the label 1 if $[f(u)f(v)]^2 \equiv 1(mod\, 3)$ and the label 0 if $[f(u)f(v)]^2 \equiv 0(mod\, 3)$ (ii) the number of edges labeled with 0 and the number of edges labeled with 1 under f^* differ by at most 1. A graph which admits a product square cordial labeling is called a product square cordial graph. In this paper we prove that the graphs such as fan, comb and crown graph are product square cordial graphs.

Keywords: *cordial labeling, product square cordial labeling, product square cordial graph.*
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1. Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [2]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by I. Cahit in [1]. Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. M. Sundaram et al. introduced the concept of product cordial labeling of a graph in [7]. A product cordial labeling of a graph G with the vertex set V is a function f from V to $\{0,1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Fermat's little theorem states that if p is a prime number and a is any integer not divisible by p, then $a^{p-1} \equiv 1\ (mod\, p)$. Motivated by the concept of product cordial labeling and Fermat's little theorem, we introduced product square cordial labeling. A *product square cordial labeling* of a graph G with vertex set V and edge set E is a bijection f from $V(G)$ to $\{1,2,\ldots,|V(G)|\}$ if there exists an induced edge function f^* from $E(G)$ to $\{0,1\}$ satisfies the following conditions. (i) For each edge $uv \in E(G)$ is assigned the label 1 if $[f(u)f(v)]^2 \equiv 1(mod\, 3)$ and the label 0 if $[f(u)f(v)]^2 \equiv 0(mod\, 3)$ (ii) the number of edges labeled with 0 and the number of edges labeled with 1 under f^* differ by at most 1. A graph which admits a product square cordial labeling is called a product square cordial graph. All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [4]. We use the following definitions in the subsequent section.

Definition 1.1: Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. The **corona** of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 1.2: [3] A **fan graph** is obtained by joining all vertices of P_n to a new vertex which is known as centre. It is denoted by F_n.

Definition 1.3: [6] A **comb graph** is obtained by joining a single pendent edge to each vertex of a path.

Definition 1.4: [5] A **crown graph** $C_n \odot K_1$ is obtained from a cycle by attaching a pendant edge to each vertex of the cycle.

2. Main Results

Theorem 2.1: The fan graph F_n is product square cordial graph for all $n \geq 2$.

Proof: Let the vertex set and edge set of F_n be $V(G) = \{ v, v_i\, ; 1 \leq i \leq n\}$ and $E(G) = \{(v\, v_i); 1 \leq i \leq n\} \cup \{(v_i v_{i+1}); 1 \leq i \leq n-1\}$. We consider the following two cases. Define $f: V(F_n) \to \{1,2,3,\dots,n+1\}$ as follows:

Case (i): If $n \equiv 0 (mod\ 3)$ or $n \equiv 1 (mod\ 3)$

$$f(v) = n + 1$$

$$f(v_i) = i \quad for\ 1 \leq i \leq n$$

The induced edge labels are

$$f^*(v_i v_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \leq i \leq n-1$$

$$f^*(v\, v_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \leq i \leq n$$

We observe that,

$$e_{f^*}(0) = n - 1, e_{f^*}(1) = n$$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 1.$

Case (ii): If $n \equiv 2 (mod\ 3)$

For $n \geq 2$,

$$f(v) = 1$$

$$f(v_1) = 2$$

$$f(v_2) = 3$$

For $n \geq 5$,

$$f(v_3) = 6$$

$$f(v_i) = \begin{cases} i+3 & for\ 3|i,\ 6 \le i \le n-2 \\ i & for\ 3 \nmid i,\ 4 \le i \le n \end{cases}$$

The induced edge labels are

$$f^*(v_i v_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \quad for\ 2 \le i \le n-1$$

$$f^*(v\, v_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3) \\ 1 & otherwise \end{cases} \quad for\ i = 1\ and\ 3 \le i \le n$$

$$f^*(v_1 v_2) = 0,\ f^*(v\, v_2) = 0$$

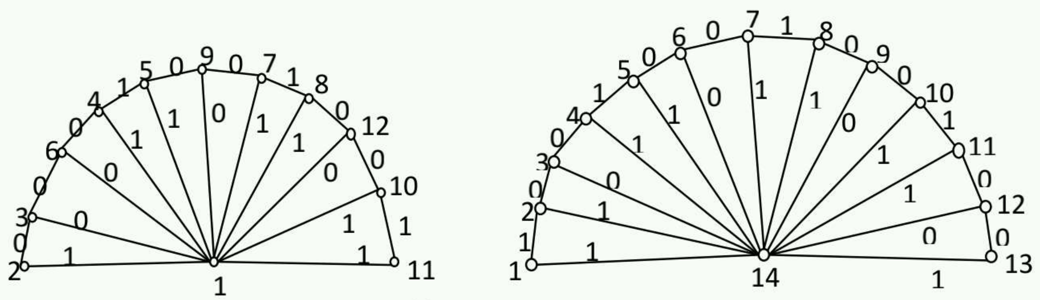
We observe that,

$$e_{f^*}(0) = n, e_{f^*}(1) = n - 1$$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

For all cases the fan F_n admits product square cordial labeling and hence the fan F_n is product square cordial graph for all $n \ge 2$.

The examples of product square cordial labeling of F_{11} and F_{13} are shown in Figure 1.



Product square cordial labeling of F_{11} Product square cordial labeling of F_{13}

Figure 1

Theorem 2.2: The comb graph $P_n \odot K_1$ is product square cordial graph for all $n \ge 2$.

Proof: Let the vertex set and edge set of $P_n \odot K_1$ be $V(P_n \odot K_1) = \{u_i, v_i\ ; 1 \le i \le n\}$ and edge set $E(P_n \odot K_1) = \{u_i u_{i+1}\ ; 1 \le i \le n-1\} \cup \{u_i v_i\ ; 1 \le i \le n\}$. We consider the following three cases.

Define $f: V(P_n \odot K_1) \to \{1,2,3, \dots, 2n\}$ as follows:

Case (i): If $n \equiv 0 (mod\ 3)$

$$f(u_i) = i \quad for\ 1 \le i \le n$$

$$f(v_i) = i + n \quad for\ 1 \le i \le n$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n-1$$

$$f^*(u_i v_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n$$

We observe that,

$$e_{f^*}(0) = n - 1, e_{f^*}(1) = n$$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

Case (ii): If $n \equiv 1 (mod\ 3)$

$$f(u_i) = i \quad for\ 1 \le i \le n$$

$$f(v_i) = 2n + 1 - i \quad for\ 1 \le i \le n$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n-1$$

$$f^*(u_i v_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n$$

We observe that,

$$e_{f^*}(0) = n - 1, e_{f^*}(1) = n$$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

Case (iii): If $n \equiv 2 (mod\ 3)$

$$f(u_i) = i \quad for\ 1 \le i \le n$$

$$f(v_i) = \begin{cases} i + n + 1 & for\ 1 \le i \le n-1 \\ n + 1 & for\ i = n \end{cases}$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n-1$$

$$f^*(u_iv_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i = n \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n$$

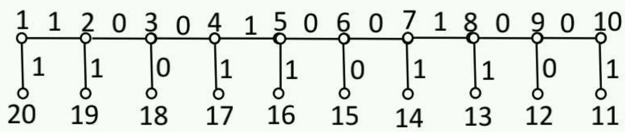
We observe that,

$$e_{f^*}(0) = n - 1, e_{f^*}(1) = n$$

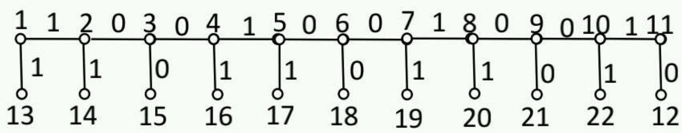
Hence $|e_{f^*}(0) - e_{f^*}(1)| = 1$.

For all cases the comb graph $P_n \odot K_1$ admits product square cordial labeling and hence the comb graph $P_n \odot K_1$ is product square cordial graph for all $n \ge 2$.

The examples of product square cordial labeling of $P_{10} \odot K_1$ and $P_{11} \odot K_1$ are shown in Figure 2.



Product square cordial labeling of $P_{10} \odot K_1$



Product square cordial labeling of $P_{11} \odot K_1$

Figure 2

Theorem 2.3: The crown graph $C_n \odot K_1$ is product square cordial graph for all $n \ge 3$.

Proof: Let the vertex set and edge set of $C_n \odot K_1$ be $V(C_n \odot K_1) = \{u_i, v_i\ ; 1 \le i \le n\}$ and edge set $E(C_n \odot K_1) = \{u_iu_{i+1}\ ; 1 \le i \le n-1\} \cup \{u_nu_1\} \cup \{u_iv_i\ ; 1 \le i \le n\}$. We consider the following three cases.

Define $f: V(C_n \odot K_1) \rightarrow \{1,2,3, \dots, 2n\}$ as follows:

Case (i): If $n \equiv 0(mod\ 3)$

$$f(u_i) = i \quad for\ 1 \le i \le n$$

$$f(v_i) = i + n \quad for\ 1 \le i \le n$$

The induced edge labels are

$$f^*(u_iu_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n-1$$

$$f^*(u_iv_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n$$

$$f^*(u_nu_1) = 0$$

We observe that,

$$e_{f^*}(0) = n, e_{f^*}(1) = n$$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 0$.

Case (ii): If $n \equiv 1(mod\ 3)$

$$f(u_i) = i \quad for\ 1 \le i \le n$$

$$f(v_i) = \begin{cases} 2n+1-i & for\ i = 1\ or\ 4 \le i \le n \\ 2n-2 & for\ i = 2 \\ 2n-1 & for\ i = 3 \end{cases}$$

The induced edge labels are

$$f^*(u_iu_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n-1$$

$$f^*(u_iv_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i = 2 \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n$$

$$f^*(u_nu_1) = 1$$

We observe that,

$$e_{f^*}(0) = n,\ e_{f^*}(1) = n$$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 0$.

Case (iii): If $n \equiv 2(mod\ 3)$

$$f(u_i) = i \quad for\ 1 \le i \le n$$

$$f(v_i) = \begin{cases} i+n & for\ 1 \le i \le 5 \\ i+n+1 & for\ 6 \le i \le n-1 \\ n+6 & for\ i = n\ and\ n \ne 5 \end{cases}$$

The induced edge labels are

$$f^*(u_iu_{i+1}) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i+1 \equiv 0 (mod\ 3) \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n-1$$

$$f^*(u_iv_i) = \begin{cases} 0 & if\ i \equiv 0\ (mod\ 3)\ or\ i = 1\ or\ i = 4 \\ 1 & otherwise \end{cases} \qquad for\ 1 \le i \le n$$

$f^*(u_n u_1) = 1$

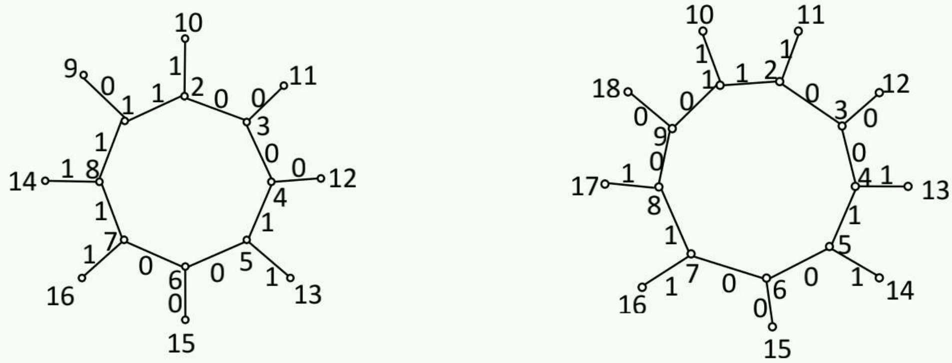
We observe that,

$e_{f^*}(0) = n, \ e_{f^*}(1) = n$

Hence $|e_{f^*}(0) - e_{f^*}(1)| = 0$.

For all cases the crown graph $C_n \odot K_1$ admits product square cordial labeling and hence the crown graph $C_n \odot K_1$ is product square cordial graph for all $n \geq 3$.

The examples of product square cordial labeling of $C_8 \odot K_1$ and $C_9 \odot K_1$ are shown in Figure 3.



Product square cordial labeling of $C_8 \odot K_1$ Product square cordial labeling of $C_9 \odot K_1$

Figure 3

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